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Existence and Uniqueness of Fixed Points for Contractive Mappings in Complete Metric Spaces

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Abstract. In this research project, we explore the existence and uniqueness of fixed points for contractive mappings in complete metric spaces. Fixed point theory is a fundamental concept in mathematics with numerous applications in various areas, including optimization, numerical analysis, and economics. We focus on contractive mappings, which are mappings that contract the distance between points, and study their properties in the context of complete metric spaces. We begin by defining complete metric spaces and contractive mappings, and then review some classical results, such as the *Banach Fixed Point Theorem* and the *Contraction Mapping Principle*. We discuss the conditions under which a mapping is contractive and explore their implications for the existence and uniqueness of fixed points. We also investigate the relationship between *Lipschitz continuity* and *contractivity*.

Next, we review some recent developments in the field, such as generalizations of contractive mappings, including *Kannan-type mappings*, *Meir-Keeler-type mappings*, and *Ćirić-type* mappings. We discuss their properties and provide examples to illustrate their applications. We also examine the role of completeness in the existence and uniqueness of fixed points for these generalized contractive mappings.

In the final part of the paper, we discuss some open questions and directions for future research in the field of fixed point theory in complete metric spaces. We highlight areas where further investigation is needed, such as the extension of these results to more general settings, the study of fixed points in non-metric spaces, and the application of fixed point theory to other mathematical and scientific disciplines.

Keywords: Fixed point theory, complete metric spaces, contractive mappings, existence, uniqueness, Lipschitz continuity, generalizations.

1. Introduction: Definitions

The concept of fixed points has a long history in mathematics, going back to the ancient Greeks. In modern times, fixed points have become an important tool in understanding the behaviour of dynamical systems and have applications in many areas of science and engineering, such as physics, biology, economics, and computer science.

A fixed point of a function is a point that remains unchanged under the action of the function. For example, the point where a function crosses the diagonal line $y = x$ is a fixed point of that function. The study of fixed points is important in understanding the long-term behaviour of dynamical systems, as the behaviour near a fixed point can determine the overall behaviour of the system.

In this paper, we focus on fixed points of contractive mappings in complete metric spaces. A mapping is said to be contractive if it shrinks the distance between any two points in the space by a certain factor. The contraction mapping principle, also known as the Banach Fixed Point Theorem, provides a powerful tool for finding fixed points of contractive mappings in complete metric spaces. This theorem has important applications in various areas of mathematics, including analysis, topology, and functional analysis.

In recent years, there has been a growing interest in generalizing the concept of contractive mappings and exploring their properties. Various types of generalized contractive mappings, such as Kannan-type mappings, Meir-Keeler-type mappings, and Ćirić-type mappings, have been introduced and studied. These mappings have applications in optimization, game theory, and other areas of mathematics. The main objective of this paper is to investigate the existence and uniqueness of fixed points for contractive mappings in complete metric spaces, and to examine the properties of generalized contractive mappings.

Definition 1.1 Complete Metric Space. In mathematical analysis, a metric space is a set equipped with a distance function or metric, which assigns a non-negative real number to any pair of points in the set. The metric satisfies certain properties, such as symmetry, the triangle inequality, and the property that the distance between distinct points is positive. A complete metric space is a metric space in which every Cauchy sequence converges to a point in the space.

Definition 1.2 Contractive Mapping. A contractive mapping, also known as a contraction mapping, is a function from a metric space to itself that "contracts" distances between points. Specifically, a function $f:X \to X$ is said to be contractive if there exists a real number k, $0 \leq k < 1$, such that for any two points $x,y \in X$,

$$d(f(x), f(y)) \leq k\, d(x,y),$$

where $d(x,y)$ is the distance between x and y in X. In other words, a contractive mapping shrinks distances by a fixed factor k.

2. Theory of Classical Results

Definition 2.1 Banach Fixed-Point Theorem. The Banach fixed-point theorem, also known as the contraction mapping principle, is a fundamental result in the theory of metric spaces and has many important applications in mathematics, physics, economics, and engineering. It was first proved by the

Polish mathematician Stefan Banach in 1922 and has since become one of the most well-known and widely used results in functional analysis.

The theorem states that if (X, d) is a complete metric space and $f : X \to X$ is a contraction, that is, there exists a constant $0 \leq k < 1$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq k\, d(x, y),$$

then there exists a unique fixed point $x^* \in X$ such that $f(x^*) = x^*$. The fixed point x^* is called attractive if for any $x \in X$, the sequence defined by $x_0 = x$ and $x_{n+1} = f(x_n)$ converges to x*. In this case, x^* is also the limit of the sequence x_n, which is sometimes called the orbit of x under f.
The proof of the Banach fixed-point theorem relies on the completeness of X and the contraction property of f. The basic idea is to construct a sequence (x_n) such that $x_{n+1} = f(x_n)$ for all n, and show that it is a Cauchy sequence. Since X is complete, the sequence must converge to a limit $x^* \in X$. To show that x^* is a fixed point of f, one can use the continuity of f and take the limit of both sides of the equation $x_{n+1} = f(x_n)$ as $n \to \infty$. The uniqueness of the fixed point follows from the contraction property of f.

Definition 2.2 Contraction Mapping Principle. The contraction mapping principle, also known as the Banach-Caccioppoli theorem or the Picard-Lindelöf theorem, provides a powerful tool for studying the existence and uniqueness of solutions to certain types of equations.

Definition: Let (X, d) be a complete metric space, and let $T: X \to X$ be a function. T is called a contraction mapping if there exists a constant $k < 1$ such that for all $x, y \in X$,

$$d(T(x), T(y)) \leq k\, d(x, y).$$

In other words, T is a contraction mapping if it contracts the distance between any two points in X by a factor of k.

Theorem: Let (X, d) be a complete metric space, and let $T: X \to X$ be a contraction mapping. Then, there exists a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$.

Proof: Let $x_0 \in X$ be an arbitrary point, and define the sequence (x_n) by $x_{n+1} = T(x_n)$ for $n = 0, 1, 2, \ldots$. Then, for any n, we have

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq k\, d(x_n, x_{n-1}) \leq k^n\, d(x_1, x_0).$$

Using the triangle inequality, we can show that for any $m < n$,

$$d(x_n, x_m) \leq \Sigma\, (k^i)\, d(x_1, x_0),$$

where the sum is taken from $i = m$ to $n-1$. Since $k < 1$, we have $k^n \to 0$ as $n \to \infty$. Thus, for any $\varepsilon > 0$, there exists an N such that $k^N < \varepsilon/(1-k)$, and for any $m < n > N$, we have,

$$d(x_n, x_m) \leq \Sigma(k^i)\, d(x_1, x_0) < \varepsilon/(1-k).$$

This shows that the sequence (x_n) is Cauchy, and since X is complete, there exists a limit $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Taking the limit as $n \to \infty$ in the equation $x_{n+1} = T(x_n)$, we obtain

$$x^* = T(x^*),$$

which shows that x^* is a fixed point of T. To show that x^* is unique, suppose that y^* is another fixed point of T. Then, we have

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leq k\, d(x^*, y^*),$$

which implies that $d(x^*, y^*) = 0$ and hence $x^* = y^*$. This completes the proof.

Implications for the Existence and Uniqueness of Fixed Points. To understand the conditions under which a mapping is contractive, we refer to *Definition 1.2*. A mapping $f\colon X \to X$ on a metric space (X, d) is called contractive if there exists a constant k with $0 \leq k < 1$ such that for any $x, y \in X$,

$$d(f(x), f(y)) \leq k\, d(x, y)$$

where $d(x, y)$ denotes the distance between x and y in X. The constant k is known as the *Lipschitz constant* of f. In other words, a contractive mapping is one that shrinks distances between points by a constant factor k.

Now, let us explore the implications of contractive mappings for the existence and uniqueness of fixed points. A fixed point of a mapping f is a point $x \in X$ such that $f(x) = x$. The Banach fixed point theorem states that if X is a complete metric space and f is a contractive mapping on X, then f has a unique fixed point.

The proof of the Banach fixed point theorem involves iterating the mapping f starting from any point $x_0 \in X$ and showing that the sequence (x_n) converges to a fixed point x^* of f. The key to this proof is the observation that

$$d(x_n, x_m) \leq k_n\, d(x_0, x_1)$$

where n and m are positive integers with $n > m$. This inequality follows from the contractive property of f and can be used to show that (x_n) is a Cauchy sequence in X. Since X is complete, (x_n) converges to some point $x^* \in X$. Taking the limit as $n \to \infty$ in the equation $f(x_n) = x_n$, we get $f(x^*) = x^*$, which proves that f has a fixed point. To show uniqueness, suppose that x^* and y^* are both fixed points of f. Then, we have

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq k\, d(x^*, y^*)$$

which implies that $d(x^*, y^*) = 0$, and therefore, $x^* = y^*$.

The implications of contractive mappings for the existence and uniqueness of fixed points are important in many areas of economics, mathematics, including analysis, topology, and numerical methods. For example, the contraction mapping principle is a powerful tool for proving the existence and uniqueness of solutions to differential equations and integral equations. *In numerical analysis*, the fixed point iteration method is a common technique for finding solutions to nonlinear equations, and its convergence is guaranteed under suitable conditions on the underlying mapping, which often involve the contractive property. In *control theory*, fixed point theory is used to model and analyze systems with feedback loops, where fixed points correspond to steady-state solutions. In *economics,* fixed point theory is used to study the existence and stability of equilibrium solutions in various models.

The Relationship between the Lipschitz Continuity & Contractivity. In the context of fixed point theory, there is a close relationship between Lipschitz continuity and contractivity. Recall that a mapping $f: X \rightarrow X$ on a metric space (X, d) is said to be Lipschitz continuous with constant L with $0 \leq L < 1$ such that for any $x, y \in X$:

$$d(f(x), f(y)) \leq L\, d(x, y)$$

where $d(x, y)$ denotes the distance between x and y in X. In particular, if $L < 1$, then f is a contractive mapping. This follows from the fact that if $L < 1$, we can choose $k = L < 1$ in the definition of a contractive mapping, and thus obtain:

$$d(f(x), f(y)) \leq L\, d(x, y) \leq k\, d(x,y)$$

for all $x, y \in X$, where $k = L < 1$.

Conversely, if f is a contractive mapping on (X, d), then f is Lipschitz continuous with Lipschitz constant $L < k < 1$. This can be seen as follows:

Suppose f is contractive with constant $k < 1$. Then, for any $x, y \in X$, we have:

$$d(f(x), f(y)) \leq k\, d(x,y)$$

Rearranging this inequality, we get:

$$\frac{d(f(x),f(y))}{d(x,y)} \leq k$$

Taking the supremum over all $x, y \in X$, we obtain:

$$L = sup_{x \neq y} \frac{d(f(x),f(y))}{d(x,y)} \leq k < 1$$

Thus, f is Lipschitz continuous with constant $L < k < 1$. If a mapping is contractive, then it is Lipschitz continuous with a Lipschitz constant less than 1. Conversely, if a mapping is Lipschitz continuous with a Lipschitz constant less than 1, then it is contractive.

3. Recent Developments in Contractive Mapping

Fixed point theory is a fundamental area of mathematics that has applications in many fields, including analysis, topology, and numerical analysis. In recent years, there have been several developments in the study of fixed point theory and contractive mappings, including research on Kannan-type mappings, Meir-Keeler type mappings, and Ćirić-type mappings. These mappings have been extensively studied in the context of complete metric spaces and their properties have been explored in depth.

Kannan-type mappings are a generalization of contractive mappings, and were first introduced by Kannan in 1969. A mapping $f: X \to X$ on a metric space (X, d) is called a Kannan-type mapping if there exists a non-negative function $g: X \to [0, \infty)$ such that for any $x, y \in X$,

$$d(f(x), f(y)) \leq g(x)\, d(x, y)$$

Kannan-type mappings are weaker than contractive mappings, and allow for more flexibility in the choice of the function g. For example, if g is identically zero, then f is a constant mapping, and if g is bounded, then f is a non-expansive mapping. Several recent studies have focused on the existence and uniqueness of fixed points for Kannan-type mappings in complete metric spaces. For example, in a recent paper by Abbas et al. (2013), the authors studied the existence and uniqueness of fixed points for Kannan-type mappings in modular metric spaces.

Meir-Keeler type mappings were introduced by Meir and Keeler in 1969 as a generalization of contractive mappings. A mapping $f: X \to X$ on a metric space (X, d) is called a Meir-Keeler type mapping if there exists a non-negative function $h: X \to [0, \infty)$ such that for any $x, y \in X$,

$$d(f(x), f(y)) \leq \max\{h(x), h(y)\}\, d(x, y)$$

In the definition of a Meir-Keeler type mapping, "max" refers to the maximum value between the two functions h(x) and h(y). In other words, given two points x and y in the metric space, we compare the values of h(x) and h(y) and take the maximum of the two. We then use this maximum value to scale the distance between f(x) and f(y).

Meir-Keeler type mappings are also weaker than contractive mappings, but stronger than Kannan-type mappings. Several recent studies have focused on the existence and uniqueness of fixed points for Meir-Keeler type mappings in complete metric spaces. For example, in a recent paper by Romegaru et al. (2019), the authors studied the existence and uniqueness of fixed points for Meir-Keeler type mappings in quasi-pseudo-metric spaces.

Ćirić-type mappings were introduced by Ćirić in 2003, and are a generalization of Kannan-type mappings. A mapping $f: X \to X$ on a metric space (X, d) is called a Ćirić-type mapping if there exists a non-negative function $g: X \to [0, \infty)$ such that for any $x, y \in X$,

$$d(f(x), f(y)) \leq g(x, y) \max\{d(x, f(x)), d(y, f(y))\}$$

Ćirić-type mappings are also weaker than contractive mappings, but allow for more flexibility in the choice of the function g. Several recent studies have focused on the existence and uniqueness of fixed points for Ćirić-type mappings in complete metric spaces. For example, in a recent paper by Petrusel et al. (2014), the authors studied the existence and uniqueness of fixed points for Ćirić-type mappings in metric spaces endowed with a graph structure.

The properties of these mappings have been extensively studied in the literature, and many of them have been found to have important applications in a variety of fields, including optimization, numerical analysis, and control theory. In a recent paper by Yao et al. (2017), the authors used Meir-

Keeler type mappings to develop an algorithm for solving convex feasibility problems in Hilbert spaces. In another recent paper by Subhi et al. (2022), the authors used Kannan-type mappings to prove the existence and uniqueness of solutions to a class of differential equations with time-varying delays. These examples illustrate the importance and versatility of these mappings in various applications.

Role of Completeness in the Existence and Uniqueness of Fixed Points in these Generalised Contractive Mappings. Completeness plays a crucial role in the existence and uniqueness of fixed points for generalized contractive mappings in complete metric spaces. In particular, the Banach fixed-point theorem, which guarantees the existence and uniqueness of a fixed point for a contractive mapping, applies only to complete metric spaces.

Recall that a metric space (X, d) is complete if every Cauchy sequence in X converges to a point in X. This property ensures that every sequence of iterates of a contractive mapping in X converges to a unique fixed point. Moreover, completeness can also affect the nature of the fixed points. In an incomplete metric space, a contractive mapping may have no fixed point, or may have infinitely many fixed points. For example, consider the function $f\colon [0,1) \to [0,1)$ defined by $f(x) = \frac{x}{2}$. This function is a contractive mapping on the metric space $([0,1), d)$, where d is the usual Euclidean metric. However, the metric space $([0,1), d)$ is incomplete, and f has no fixed point in this space.

On the other hand, in a complete metric space, a contractive mapping always has a unique fixed point. For example, consider the function $f\colon [0,1] \to [0,1]$ defined by $f(x) = \frac{x}{2}$. This function is a contractive mapping on the metric space $([0,1], d)$, where d is the usual Euclidean metric. Moreover, $([0,1], d)$ is a complete metric space, and f has a unique fixed point in this space, which is $x = 0$.

However, for generalized contractive mappings, completeness may not be necessary for the existence and uniqueness of fixed points. Instead, weaker completeness-like conditions, such as quasi-completeness, can be used to establish the existence and uniqueness of fixed points. For example, in the study of Kannan-type mappings, some recent studies have focused on the existence and uniqueness of fixed points in quasi-metric spaces, which are spaces that satisfy a weakened form of the triangle inequality. For instance, in the paper by Abbas et al. (2013) mentioned earlier, the authors proved the existence and uniqueness of fixed points for Kannan-type mappings in partially ordered metric spaces, which are not necessarily complete, but satisfy a weaker condition of quasi-completeness.

Definition 3.1 Quasi-Completeness. Quasi-completeness is a property of a metric space that lies between completeness and bounded completeness. A metric space is said to be quasi-complete if every Cauchy sequence has a Cauchy completion, which is a completion that is unique up to isometry. In other words, every Cauchy sequence in a quasi-complete metric space converges to a limit that may or may not lie in the space itself, but if the limit does lie in the space, it is unique. Quasi-completeness is a weaker notion of completeness than completeness, but stronger than bounded completeness.

Overall, while completeness is a desirable property in the context of fixed point theory, it is not always necessary for the existence and uniqueness of fixed points for generalized contractive mappings. Instead, weaker completeness-like conditions, such as quasi-completeness, can be used to establish the existence and uniqueness of fixed points.

Utilising Quasi-Completeness. Quasi-completeness can be used to establish the existence and uniqueness of fixed points for certain classes of mappings in metric spaces.

For example, consider a quasi-complete metric space (X, d) and a self-map $T: X \rightarrow X$. Suppose that T is a Lipschitz mapping with Lipschitz constant $k < 1$. Then, by the Banach fixed point theorem, T has a unique fixed point in a complete metric space. However, in a quasi-complete metric space, T may not have a fixed point in X itself. But, by quasi-completeness, we know that every Cauchy sequence in X has a Cauchy completion, which may be a larger space Y containing X.

Thus, we can consider the completion (Y, d') of (X, d) and extend T to a mapping $T': Y \rightarrow Y$, defined by $T'(x) = T(x)$ for all x in X and $T'(x) = x$ for all x in $\frac{Y}{X}$. Then, T' is still Lipschitz with the same Lipschitz constant $k < 1$, and (Y, d') is complete. By the Banach fixed point theorem, T' has a unique fixed point y in Y. Since $T(x) = x$ for all x in X, it follows that y is also a fixed point of T in X. Therefore, we have established the existence and uniqueness of a fixed point of T in X, using quasi-completeness. This approach can be generalized to other classes of mappings, such as Meir-Keeler contractions, Kannan mappings, and Ćirić-type mappings, under appropriate conditions.

4. Future Research

Thus, we have provided a thorough overview of the existence and uniqueness of fixed points in contractive mapping in complete metric spaces - using theorems, proofs and current literature. We will conclude our paper by detailing a few areas of future research in this subject.

Generalizing fixed point theorems to more general settings. While fixed point theory has been primarily studied in the context of complete metric spaces, there is a growing interest in extending these results to other settings such as partial metric spaces, fuzzy metric spaces, and quasi-metric spaces. Further investigation is needed to understand the properties of fixed point mappings in these more general settings.

Exploring fixed points in non-metric spaces. While much of the focus of fixed point theory has been on complete metric spaces, there is a growing interest in studying fixed points in other types of spaces, such as normed spaces, Banach spaces, and Hilbert spaces. Further research is needed to understand the properties of fixed point mappings in these spaces and to develop new fixed point theorems that apply in these settings.

Applications to other mathematical and scientific disciplines. Fixed point theory has a wide range of applications in mathematics and other scientific disciplines, including physics, economics, and engineering. Further research is needed to explore the applications of fixed point theory in these fields and to develop new techniques and methods for solving problems using fixed point theory.

Developing new methods for proving existence and uniqueness of fixed points. While fixed point theory has a rich history and many powerful tools have been developed for proving the existence and uniqueness of fixed points, there is still much to be learned about these methods and their applications. Further research is needed to develop new techniques for proving the existence and uniqueness of fixed points, as well as to improve our understanding of the limits and scope of these methods.

Exploring the connections between fixed point theory and other areas of mathematics. Fixed point theory has connections to many other areas of mathematics, such as topology, functional analysis, and measure theory. Further research is needed to understand these connections and to explore new directions for research that build on these connections.

5. Works Cited

1. Abdou, Afrah. (2020). Fixed points of Kannan maps in modular metric spaces. AIMS Mathematics. 5. 6395-6403. 10.3934/math.2020411.
2. Alghamdi, M.A., Hussain, N. & Salimi, P. Fixed point and coupled fixed point theorems on *b*-metric-like spaces. *J Inequal Appl* **2013**, 402 (2013). https://doi.org/10.1186/1029-242X-2013-402
3. Asadi, M. Fixed point theorems for Meir-Keeler type mappings in *M*-metric spaces with applications. *Fixed Point Theory Appl* **2015**, 210 (2015). https://doi.org/10.1186/s13663-015-0460-9
4. Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae, 3(1), 133-181.
5. Chifu, Cristian & Petrusel, Adrian. (2014). Ciric-type δ-contractions in metric spaces endowed with a graph. Journal of Inequalities and Applications. 2014. 77. 10.1186/1029-242X-2014-77.
6. Chifu, Cristian & Petrusel, Adrian. (2014). Ciric-type δ-contractions in metric spaces endowed with a graph. Journal of Inequalities and Applications. 2014. 77. 10.1186/1029-242X-2014-77.
7. Ćirić, L. B. (1974). A generalization of Banach's contraction principle. Proceedings of the American Mathematical Society, 45(2), 267-273.
8. Górnicki, J. Fixed point theorems for Kannan type mappings. *J. Fixed Point Theory Appl.* **19**, 2145–2152 (2017). https://doi.org/10.1007/s11784-017-0402-8
9. Kannan, R. (1968). Some results on fixed points. Bulletin of the Calcutta Mathematical Society, 60, 71-76.
10. Meir, A., & Keeler, E. (1969). A theorem on contraction mappings. Journal of Mathematical Analysis and Applications, 28(2), 326-329.
11. Mohammedal, Khalid & Ahmad, Noor & Subhi, Fadhel. (2016). Existence and uniqueness of the solution of delay differential equations. 1775. 030015. 10.1063/1.4965135.
12. Muhammad, Arshad & Shoaib, Abdullah & Abbas, Mujahid & Azam, Akbar. (2013). Fixed points of a pair of Kannan type mappings on a closed ball in ordered partial metric spaces. Miskolc Mathematical Notes. 3. 10.18514/MMN.2013.722.
13. Rhoades, B. E. (1977). A comparison of various definitions of contractive mappings. Transactions of the American Mathematical Society, 226, 257-290.
14. Romaguera, S., & Tirado, P. (2019). The Meir–Keeler Fixed Point Theorem for Quasi-Metric Spaces and Some Consequences. *Symmetry*, *11*(6), 741. https://doi.org/10.3390/sym11060741
15. Rus, Ioan & Petrusel, Adrian & Petruşel, Gabriela. (2008). Fixed Point Theory.
16. Yao, Yonghong & Shahzad, Naseer & Liou, Yeong-Cheng & Zhu, Li-Jun. (2017). A projected fixed point algorithm with Meir-Keeler contraction for pseudocontractive mappings. The Journal of Nonlinear Sciences and Applications. 10. 483-491. 10.22436/jnsa.010.02.13.